Solutions to *Variational Analysis* by Rockafellar and Wets

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A quick test

1 1D

1.12 A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is continuous if and only if it is both lower semicontinuous and upper semicontinuous.

By the definition of the lower limit

$$\liminf_{x \to \overline{x}} f(x) = \lim_{\delta \to 0} \left[\inf_{x \in \mathbb{B}\overline{x},\delta} f(x) \right].$$

Using the definition of the limit (see, for example Bartle and Sherbert's *Introduction to Real Analysis*) this means

$$\forall \epsilon \exists \delta_0 > 0 \text{ such that } \delta < \delta_0 \Rightarrow \left| \inf_{x \in \mathbb{B}(\overline{x},\delta)} f(x) - \liminf_{x \to \overline{x}} f(x) \right| < \epsilon.$$

Note that $\inf_{x \in B(\overline{x},\delta)} f(x)$ is increasing in δ so that this last expression is equivalent to

$$\liminf_{x \to \overline{x}} f(x) - \inf_{x \in \mathbb{B}(\overline{x},\delta)} f(x) < \epsilon.$$

Similarly, the definition of upper limit means

$$\forall \epsilon \exists \delta_1 > 0 \text{ such that } \delta < \delta_1 \Rightarrow \sup_{x \in \mathbb{B}(\overline{x},\delta)} f(x) - f(\overline{x}) < \epsilon.$$

Now we will complete the problem. Assume that f is both lower and upper semicontinuous. Note that each of the implications that follow are reversible, so that the reverse direction also follows.

To show that f is continuous at $\overline{x}$, we need to show that $\forall \epsilon > 0 \exists \delta_2$ such that $x \in \mathbb{B}(\overline{x}, \delta) \Rightarrow |f(x) - f(\overline{x})| < \epsilon$. Fix $\epsilon > 0$. Take $\delta = \min\{\delta_0, \delta_1\}$, where $delta_0$ and δ_1 come from the definitions of lower and upper semicontinuity. We then have, for $x \in B(\overline{x}, \delta)$,

$$
\begin{aligned}
&|f(x) - f(\overline{x})| \\
&= \max\{f(x) - f(\overline{x}), f(\overline{x}) - f(x)\} \\
&\leqslant \max\{\sup_{x \in \mathbb{B}(\overline{x},\delta)} f(x) - f(\overline{x}), \sup_{x \in \mathbb{B}(\overline{x},\delta)} f(\overline{x}) - f(x)\} \\
&\max\{\sup_{x \in \mathbb{B}(\overline{x},\delta)} f(x) - f(\overline{x}), f(\overline{x} - \inf_{x \in \mathbb{B}(\overline{x},\delta)} f(x)\} \\
&< \max\{\epsilon, \epsilon\} = \epsilon.
\end{aligned}
$$

So we are done. Note that splitting the absolute value into the maximum of its positive and negative parts is the crux of this problem, and allows us to incorporate the definitions of lower and upper semicontinuity. So"controlling" both of these expressions gives us continuity.

1.13 For an arbitrary function $f : \mathbb{R}^n \to \overline{R}$ and a pair of elements $\overline{x} \in \mathbb{R}^n$ and $\overline{\alpha} \in \mathbb{R}$, one has

a $(\overline{x}, \overline{\alpha}) \in \text{cl}(\text{epi} f)$ if and only if $\overline{\alpha} \geqslant \liminf_{x \to \overline{x}} f(x)$.

Assume $(\overline{x}, \overline{\alpha}) \in \text{cl}(\text{epi} f)$. Then there is a sequence $(x^\nu, \alpha^\nu) \to (\overline{x\alpha})$ with $\alpha^\nu \geqslant f(x^\nu)$ for all $\nu \in \mathbb{N}$. Taking $\liminf_{\nu \to \infty}$ of both sides of this inequality gives

$$\overline{\alpha} \geqslant \liminf_{n \to \infty} f(x^\nu)$$

this last expression is greater than or equal to $\liminf_{x \to \overline{x}} f(x)$ by Lemma 1.17

For the reverse direction, if $\overline{\alpha} \geqslant \liminf_{x \to \overline{x}} f(x)$, let x^ν by a sequence converging to $\overline{x}$ such that $\lim_{n \to \infty} f(x^\nu) \to \liminf_{x \to \overline{x}}$. Such a sequence exists by Lemma 1.17. Then consider the sequence $(x^\nu, f(x^\nu) + \alpha - \liminf_{x \to \overline{x}} f(x))$. Since $\overline{\alpha} \geqslant \liminf_{x \to \overline{x}} f(x)$, this is in epi$f$. This clearly converges to $(\overline{x}, \overline{\alpha})$, so that $(\overline{x}, \overline{\alpha}) \in \text{cl}(\text{epi} f)$.

b $(\overline{x}, \overline{\alpha}) \in \text{int}(\text{epi} f)$ if and only if $\overline{\alpha} > \limsup_{x \to \overline{x}}$.

For ease of computation in this problem, we'll equip $\mathbb{R}^{n+1}$ with the one-norm topology. In other words,

$$d(x, y) = \sum_{i=1}^{n+1} |x_i - y_i|.$$

Because R^{n+1} is finite dimensional, all norms on it are equivalent and hence we have this liberty. See, for example, Applied Analysis by Hunter Nachtergale, section 5.4.

We will prove the negation of this statement. Assume that $\overline{\alpha} \leqslant \limsup_{x\to\overline{x}} f(x)$. Take $x^\nu \to \overline{x}$ such that

$$\lim_{\nu\to\infty} f(x^\nu) = \limsup_{x\to\overline{x}} f(x).$$

Such a sequence exists by Lemma 1.7 applied to lim sup. We need to show that $(\overline{x}, \overline{\alpha}) \notin \text{int}(\text{epi}f)$, i.e. for every $\epsilon > 0$, $\mathbb{B}((\overline{x}, \overline{\alpha}), \epsilon) \nsubseteq \text{epi}(f)$. Fix $\epsilon > 0$. Choose N so that $\nu > N$ implies

$$d((x^\nu, f(x^\nu) - (\limsup_{x\to\overline{x}} f(x) - \overline{\alpha})), (\overline{x}, \overline{\alpha})) < \epsilon/2.$$

This can be done because by our choice of x^ν, $f(x^\nu) \to \limsup_{x\to\overline{x}} f(x)$.

Consider the sequence

$$(x^\nu, f(x^\nu) - (\limsup_{x\to\overline{x}} f(x) - \overline{\alpha}) - \epsilon/2).$$

This is not in epi$f$ because $\overline{\alpha} \leqslant \limsup_{x\to\overline{x}} f(x)$. Then, because we work with the 1-norm, for $\nu > N$,

$$d((x^\nu, f(x^\nu) - (\limsup_{x\to\overline{x}} f(x) - \overline{\alpha}) - \epsilon/2), (\overline{x}, \overline{\alpha}))$$

$$\leqslant ||x^\nu - \overline{x}||_1 + \left| f(x^\nu) - (\limsup_{x-\overline{x}} f(x) - \overline{\alpha}) - \overline{\alpha} - \epsilon/2 \right|$$

$$\leqslant ||x^\nu - x||_1 + \left| f(x^\nu - \limsup_{x\to\overline{x}} f(x) \right| + \epsilon/2$$

$$= d((x^{nu}, f(x^\nu) - (\limsup_{x\to\overline{x}} f(x) - \overline{\alpha})), (\overline{x}, \overline{\alpha})) + \epsilon/2$$

$$< \epsilon.$$

So for $\nu > N$, $(x^\nu, f(x^\nu) - (\limsup_{x\to\overline{x}} f(x) - \overline{\alpha}) - \epsilon/2)$ is in the ϵ ball of $(\overline{x}, \overline{\alpha})$, but not in the epigraph. We conclude that $(\overline{x}, \overline{\alpha}) \notin \text{int}(\text{epi}f)$.

For the reverse direction, assume $(\overline{x}, \overline{\alpha}) \notin \text{int}(\text{epi}f)$. Then for every $\epsilon > 0$ there exists (x, α) such that $(x, \alpha) \in B((\overline{x}, \overline{\alpha}), \epsilon)$ and not in epi$f$. Build a sequence by taking $\epsilon = 1/\nu$ and choosing a corresponding (x^ν, α^ν). Then since each of these points are not in the epigraph of f,

$$\alpha^\nu < f(x^\nu)$$

Taking limits of this expression

$$\overline{alpha} \leqslant \lim_{\nu\to\infty} f(x^\nu) \leqslant \limsup_{x\to\overline{x}} f(x)$$

where the last inequality follows from the characterization of lim sup as the maximum limit point (see Lemma 1.7 for lim sup).

c $(\overline{x}, \overline{\alpha}) \notin \text{cl}(\text{epi}f)$ if and only if $(\overline{x}, \overline{\alpha}) \in \text{int}(\text{hypo}f)$.

Note that $(\overline{x}, \overline{\alpha}) \in \text{epi} - f$ if and only if $(\overline{x}, -\overline{\alpha}) \in \text{hypo}f$. This follows from opening up the definitions and the fact that

$$\alpha \geqslant \liminf_{x \to \overline{x}} -f(x) \iff -\alpha \leqslant \limsup_{x \to \overline{x}} f(x).$$

Thus we have

$$\begin{aligned} (\overline{x}, \overline{\alpha}) \notin \text{cl}(\text{epi}f) &\underbrace{\iff}_{\text{part (a)}} \overline{\alpha} < \liminf_{x \to \overline{x}} f(x) \\ &\iff -\alpha > \limsup_{x \to \overline{x}} -f(x) \\ &\underbrace{\iff}_{\text{part (b)}} (\overline{x}, -\overline{\alpha}) \in \text{int}(\text{epi} - f) \\ &\underbrace{\iff}_{\text{comment above}} (\overline{x}, \overline{\alpha}) \in \text{hypo}f \end{aligned}$$

d $(\overline{x}, \overline{\alpha}) \notin \text{int}(\text{epi}f)$ if and only if $(\overline{x}, \overline{\alpha}) \in \text{cl}(\text{hypo}f)$.

Similiar to (c), we have

$$\begin{aligned} (\overline{x}, \overline{\alpha}) \notin \text{int}(\text{epi}f) &\underbrace{\iff}_{\text{by (b)}} \overline{\alpha} \leqslant \limsup_{x \to \overline{x}} f(x) \\ &\iff -\overline{\alpha} \geqslant \liminf_{x \to \overline{x}} -f(x) \\ &\underbrace{\iff}_{\text{by (a)}} (\overline{x}, \overline{\alpha}) \in \text{cl}(\text{epi} - f) \\ &\iff (\overline{x}, \overline{\alpha}) \in \text{cl}(\text{hypo}f). \end{aligned}$$

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