# Duality in Extended Linear-Quadratic Estimation and Control

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# Dynamics and Observation Model

Process:

$$x_{t+1} = F_t x_t + w_t$$

Measurements:

$$z_t = H_t x_t + v_t$$

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- $w \in \mathbb{R}^n$, random noise
- $z \in \mathbb{R}^m$, an observation
- $v \in \mathbb{R}^m$, more noise
- $H$ and $F$ are real-valued matrices.

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**Problem**

Given the observed values of $z_0, ..., z_t$, find an estimate of $x_T$ which minimizes some loss function.

# Introducing the Problem

Possible cases.

1. $T < t$. **Data Smoothing Problem**: Estimate previous state from current measurements.
2. $T = t$. **Filtering Problem**: Sequential Estimation of states.
3. $T > t$. **Prediction Problem**: Estimate future state from current measurements.

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**Example to keep in mind: Autopilot in UAVs.**

A drone is equipped with a GPS and accelerometer. Where was it, where is it, and where will it go?

# The Classical Case

## Classical Assumptions

- Let $\{w_t\}$, $\{v_t\}$ be Gaussian with mean zero.
- The error functional is the distribution of $x_T | z_0, ..., z_t$.
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Considered by Rudolf Kalman in 1960, who used these assumptions to derive the **Kalman Filter**.

- Recursive estimator: only keeps track of *a posteriori* state estimate and covariance matrix
- Requires only matrix multiplication
- Ubiquitous in practice

# Mathematical Structure

Central to Kalman's derivation is the following theorem:

**Theorem (Kalman, 1960)**

*The classical MAP problem is dual to the Linear-Quadratic Regulator problem of optimal control, in the sense that there is a bijection between the Riccati equations that characterize their solutions.*

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- A Riccati equation is a matrix equation for $P_t$, where

$$x_t = P_t z_t$$

- Satisfies the conditions for optimality derived from the problem's Hamiltonian.
- Used to translate problem from one domain (estimation) into another (optimal control).

# Mathematical Structure

**Theorem (Duality Correspondence)**

*The Riccati equation of the filter for the system*

$$\begin{aligned} x_{t+1} &= F_t x_t + w_t, & z_t &= H_t x_t + v_t, \\ w_t &\sim \mathcal{N}(0, \mathcal{P}_t), & v_t &= 0 \end{aligned}$$

*is the same as that for the linear regulator of the system*

$$y_{t-1} = F_t' y_t + H_t' u_t,$$

*with cost rate*

$$y_t' P_t y_t$$

# Extensions

A number of extensions have been made to this duality of estimation and control.

**Theorem (Todorov 2008)**

*A control problem with dynamics*

$$y_{t+1} = a_t(y_t) + u_t$$

*and cost rate*

$$q_t(z_t) + k_t(a_t(y_t) + u_t)$$

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Here, duality is shown by creating a bijection between the *Hamiltonian-Jacobi-Bellman* equations which characterize the solutions of each problem.

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## Theorem (Simon and Stubberud, 1970)

*The smoothing problem, with observations $T > t$, given by*

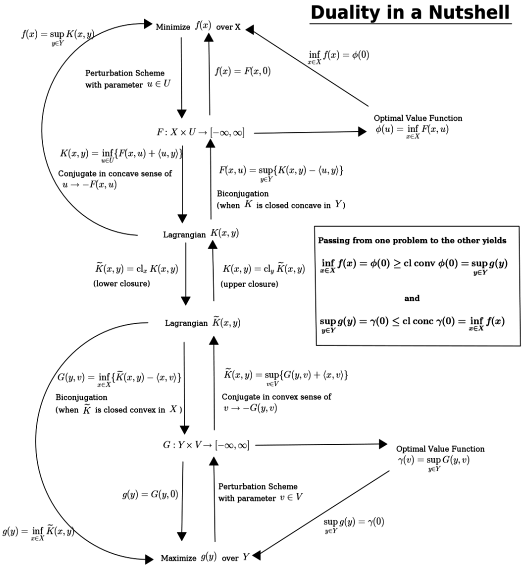
$$x_{t+1} = F_t x_t + w_t \qquad z_t = H_t x_t + v_t \quad x_0 = w_0$$
$$w_t \sim \mathcal{N}(0, \mathcal{P}_t) \qquad v_t \sim \mathcal{N}(0, \mathcal{Q}_t)$$

*is dual (in the convex-analytic sense) to the LQR problem*

$$y_{t-1} = F_t' y_t + H_t' u_t, \quad y_T = H_T' u_T$$

$$\text{with cost } \sum_{t=0}^{T} \frac{1}{2} x_t' P_t x_t + \frac{1}{2} u_t' Q_t u_t - z_t' u_t$$

# Convex Analytic Duality



# PLQ Functions

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## Definition

A *piecewise linear-quadratic function* (Rockafellar, Wets '98) is a function $\rho : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ of the form

$$\rho_{U,M}(y) = \sup_{u \in U} \{\langle u, y \rangle - \frac{1}{2}\langle u, Mu \rangle\}$$

where $U \subset \mathbb{R}^n$ is polyhedral and $M \succeq 0$

- The function $\rho(y)$ is said to be *coercive* if $\lim_{||y|| \to \infty} \rho(y) = \infty$.

# PLQ Functions

PLQ functions are attractive for a number of reasons

1. Very general framework for penalty functions
    - Hard Constraints
    - $\ell_1$ penalty
    - $\ell_2$ penalty
    - Elastic net penalty
    - Huber penalty
    - Vapnik penalty
2. Possess a common structure amenable to computation (Aravkin, Burke, Pilloneto 2013).

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If a PLQ function $\rho$ is coercive, we can use it to define a density $p(y) \propto \exp[-\rho(y)]$.

# Example

Huber penalty:

$$L_\delta(x) = \begin{cases} \frac{1}{2}x^2 & |x| < \delta \\ \delta(|x| - \frac{1}{2}\delta) & \text{otherwise} \end{cases}$$

Taking $U = [-\delta, \delta]$ and $M = I$ in the PLQ definition gives the Huber penalty.

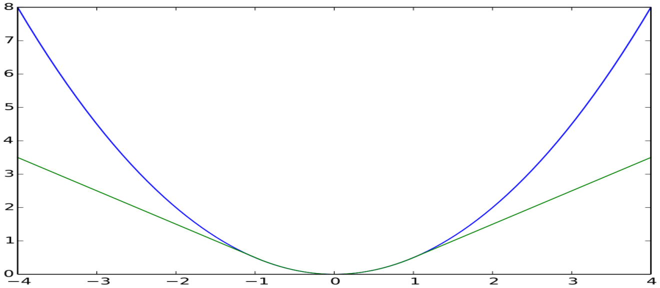


Figure : Huber loss in green, quadratic in blue

# Extended Linear-Quadratic Programming

A control problem of the form

$$\sum_{t=0}^{T} \frac{1}{2} y_t' P_t y_t + p_t' y_t + \rho_{t,U_t,M_t}(u_t)$$

$$y_{t+1} = A_t y_t + B_t u_t, \quad y_0 = A_0 u_0$$

is called an *Extended Linear-Quadratic Program*.

- Introduced in context of deterministic and stochastic control by Rockafellar and Wets in '90.
- Applied to Hydropower scheduling problem by Salinger '97.
- Versatile, still retains attractive computational structure.

# Results

## Theorem (B., Casey, Wets)

*If $w_t$ and $v_t$ have a PLQ density*

$$w_t \propto \exp[-\rho_{t,W_t,M_t}(y)] \quad v_t \propto \exp[-\rho_{t,V_t,N_t}(y)]$$

*with $M, N \succeq 0$, then the smoothing MAP problem with dynamics*

$$x_{t+1} = F_t x_t + w_t, \quad z_t = H_t x_t + v_t$$

*is dual in the convex analytic sense to the control problem*

$$y_t = F_t' y_{t+1} + H_t' u_t, \;\; y_T = H_T' v_T$$

$$y_t \in W_t, \;\; u_t \in V_t$$

$$\text{with cost} \sum_{t=0}^{T} \frac{1}{2} y_t' M_t y_t + \frac{1}{2} u_t' N_t u_t - z_t' u_t$$

When $N \succ 0$, control objective becomes

$$\sum_{t=0}^{T} \frac{1}{2} y_t' M_t y_t + \frac{1}{2}(u_t - N_t^{-1} z_t)' N (u_t - N_t^{-1} z_t)$$

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- System state near 0
- Control follows trajectory $\{N^{-1}(z_t)\}_{t=T}^{0}$

This result generalizes the classical case, because the normal distribution is a PLQ density.

# Application to Nonparametric Estimation

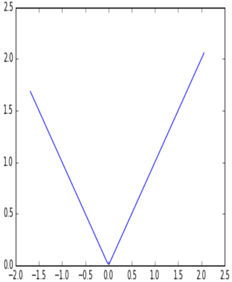



Figure : Laplace Penalty

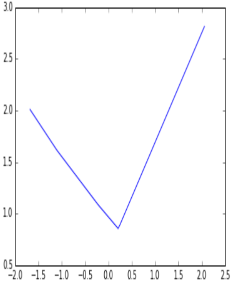



Figure : Nonparametric Estimation of Laplace Penalty $\rho$

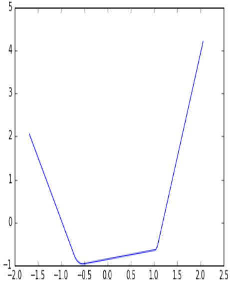
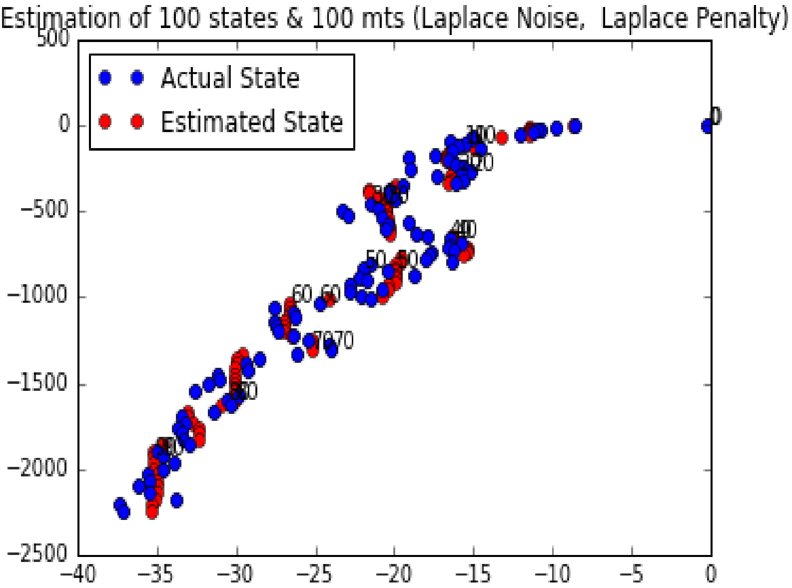
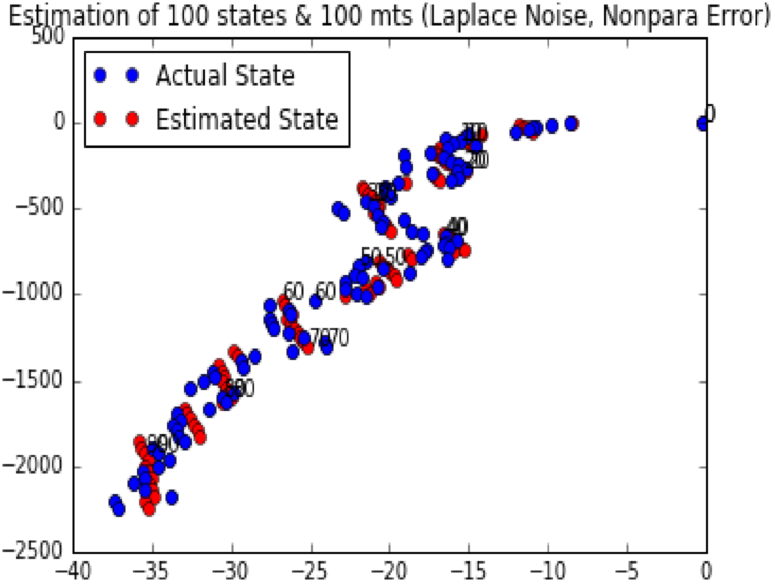



Figure : Conjugate of Nonparametric Estimation $\rho^*$

# Performance of Laplacian MAP



# Performance of Data-derived MAP



# Conclusion

- Want to generalize classical estimation results to the case of PLQ case
- Attractive computational results exist when noise is assumed to be normal.
  - General framework that allows for diverse range of distributions
  - Still contains structure similar to quadratic case
- Estimation and Control Duality still holds in PLQ setting
- How can this structure be used to our advantage while performing computations?

Thank you for your attention!

For more information see

*Log-Concave Duality in Estimation and Control* (Working Paper).
Bassett, Casey, Wets. '16

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