Duality in Extended Linear-Quadratic Estimation and Control

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March 25th, 2016
SMAI-MODE 2016, Toulouse
Dynamics and Observation Model

Process:

\[ x_{t+1} = F_t x_t + w_t \]

Measurements:

\[ z_t = H_t x_t + v_t \]
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- \( x \in \mathbb{R}^n \), a state vector
- \( w \in \mathbb{R}^n \), random noise
- \( z \in \mathbb{R}^m \), an observation
- \( v \in \mathbb{R}^m \), more noise
- \( H \) and \( F \) are real-valued matrices.
Introducing the Problem

Dynamics and Observation Model

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Problem

Given the observed values of \( z_0, \ldots, z_t \), find an estimate of \( x_T \) which minimizes some loss function.
Introducing the Problem

Possible cases.

1. $T < t$. **Data Smoothing Problem**: Estimate previous state from current measurements.

2. $T = t$. **Filtering Problem**: Sequential Estimation of states.

3. $T > t$. **Prediction Problem**: Estimate future state from current measurements.

Example to keep in mind: Autopilot in UAVs. A drone is equipped with a GPS and accelerometer. Where was it, where is it, and where will it go?
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The Classical Case

Classical Assumptions

- Let $\{w_t\}, \{v_t\}$ be Gaussian with mean zero.
- The error functional is the distribution of $x_T | z_0, ..., z_t$.
- This gives the Maximum A Posteriori (MAP) Estimator of $x_T$ given $z$. 
Introducing the Problem

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Considered by Rudolf Kalman in 1960, who used these assumptions to derive the **Kalman Filter**.

- Recursive estimator: only keeps track of a posteriori state estimate and covariance matrix
- Requires only matrix multiplication
- Ubiquitous in practice
Central to Kalman’s derivation is the following theorem:

**Theorem (Kalman, 1960)**

The classical MAP problem is dual to the Linear-Quadratic Regulator problem of optimal control, in the sense that there is a bijection between the Riccati equations that characterize their solutions.
Central to Kalman’s derivation is the following theorem:

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*The classical MAP problem is dual to the Linear-Quadratic Regulator problem of optimal control, in the sense that there is a bijection between the Riccati equations that characterize their solutions.*

A Riccati equation is a matrix equation for $P_t$, where

$$x_t = P_t z_t$$

- Satisfies the conditions for optimality derived from the problem’s Hamiltonian.
- Used to translate problem from one domain (estimation) into another (optimal control).
Introducing the Problem

Mathematical Structure

Theorem (Duality Correspondence)

The Riccati equation of the filter for the system

\[ x_{t+1} = F_t x_t + w_t, \quad z_t = H_t x_t + v_t, \]
\[ w_t \sim \mathcal{N}(0, \mathcal{P}_t), \quad v_t = 0 \]

is the same as that for the linear regulator of the system

\[ y_{t-1} = F'_t y_t + H'_t u_t, \]

with cost rate

\[ y'_t P_t y_t \]
A number of extensions have been made to this duality of estimation and control.

**Theorem (Todorov 2008)**

*A control problem with dynamics*

\[ y_{t+1} = a_t(y_t) + u_t \]

*and cost rate*

\[ q_t(z_t) + k_t(a_t(y_t) + u_t) \]

*has a dual estimation problem, where* \( w_t \propto e^{-k_t} \), \( v_t \propto e^{-q_t} \).*
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Here, duality is shown by creating a bijection between the *Hamiltonian-Jacobi-Bellman* equations which characterize the solutions of each problem.
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**Theorem (Simon and Stubberud, 1970)**

The smoothing problem, with observations $T > t$, given by

\[
\begin{align*}
  x_{t+1} &= F_t x_t + w_t \\
  z_t &= H_t x_t + v_t \\
  x_0 &= w_0 \\
  w_t &\sim \mathcal{N}(0, P_t) \\
  v_t &\sim \mathcal{N}(0, Q_t)
\end{align*}
\]

is dual (in the convex-analytic sense) to the LQR problem

\[
\begin{align*}
  y_{t-1} &= F_t' y_t + H_t' u_t, \\
  y_T &= H_T' u_T \\
  \text{with cost} \quad \sum_{t=0}^{T} \frac{1}{2} x_t' P_t x_t + \frac{1}{2} u_t' Q_t u_t - z_t' u_t
\end{align*}
\]
Introducing the Problem

Convex Analytic Duality

Duality in a Nutshell

\[ f(x) = \sup_{y \in Y} K(x, y) \]

Minimize \( f(x) \) over \( X \)

Optimal Value Function
\[ \phi(u) = \inf_{x \in X} F(x, u) \]

\[ \inf_{x \in X} f(x) = \phi(0) \]

\[ f(x) = F(x, 0) \]

Perturbation Scheme with parameter \( u \in U \)

\[ F: X \times U \to [-\infty, \infty] \]

\[ K(x, y) = \inf_{u \in U} \{ F(x, u) + \langle u, y \rangle \} \]

Conjugate in concave sense of \( u \to F(x, u) \)

\[ F(x, u) = \sup_{y \in Y} \{ K(x, y) - \langle u, y \rangle \} \]

Biconjugation (when \( K \) is closed concave in \( Y \))

\[ \phi(u) = \inf_{x \in X} F(x, u) \]

Lagrangian
\[ K(x, y) \]

\[ \bar{K}(x, y) = \text{cl}_K K(x, y) \]

(lower closure)

\[ K(x, y) = \text{cl}_y \bar{K}(x, y) \]

(upper closure)

Lagrangian
\[ \bar{K}(x, y) \]

\[ G(y, v) = \inf_{x \in X} \{ \bar{K}(x, y) - \langle x, v \rangle \} \]

Biconjugation (when \( K \) is closed convex in \( X \))

\[ \bar{K}(x, y) = \sup_{y \in Y} \{ G(y, v) + \langle x, v \rangle \} \]

Conjugate in convex sense of \( v \to -G(y, v) \)

\[ G: Y \times V \to [-\infty, \infty] \]

\[ g(y) = G(y, 0) \]

\[ g(y) = \inf_{x \in X} \bar{K}(x, y) \]

Maximize \( g(y) \) over \( Y \)

Optimal Value Function
\[ \gamma(v) = \sup_{y \in Y} G(y, v) \]

\[ \sup_{y \in Y} g(y) = \gamma(0) \]

\[ g(y) = \inf_{x \in X} \bar{K}(x, y) \]

Perturbation Scheme with parameter \( v \in V \)

Passing from one problem to the other yields
\[ \inf_{x \in X} f(x) = \phi(0) \geq \text{cl conv } \phi(0) = \sup_{y \in Y} g(y) \]

and

\[ \sup_{y \in Y} g(y) = \gamma(0) \leq \text{cl conc } \gamma(0) = \inf_{x \in X} f(x) \]
Does Convex-Analytic duality have an extension similar to Todorov’s?
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**Definition**

A *piecewise linear-quadratic function* (Rockafellar, Wets ‘98) is a function \( \rho : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\} \) of the form

\[
\rho_{U,M}(y) = \sup_{u \in U} \{ \langle u, y \rangle - \frac{1}{2} \langle u, Mu \rangle \}
\]

where \( U \subset \mathbb{R}^n \) is polyhedral and \( M \succeq 0 \)

- The function \( \rho(y) \) is said to be **coercive** if \( \lim_{\|y\| \to \infty} \rho(y) = \infty \).
PLQ functions are attractive for a number of reasons

1. Very general framework for penalty functions
   - Hard Constraints
   - $\ell_1$ penalty
   - $\ell_2$ penalty
   - Elastic net penalty
   - Huber penalty
   - Vapnik penalty

2. Possess a common structure amenable to computation (Aravkin, Burke, Pilloneto 2013).
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2. **Possess a common structure amenable to computation** (Aravkin, Burke, Pilloneto 2013).

   If a PLQ function $\rho$ is coercive, we can use it to define a density
   
   $$p(y) \propto \exp[-\rho(y)].$$
Introducing the Problem

Example

Huber penalty:

\[ L_\delta(x) = \begin{cases} 
\frac{1}{2}x^2 & |x| < \delta \\
\delta(|x| - \frac{1}{2}\delta) & \text{otherwise}
\end{cases} \]

Taking \( U = [-\delta, \delta] \) and \( M = I \) in the PLQ definition gives the Huber penalty.

Figure: Huber loss in green, quadratic in blue
Introducing the Problem

Extended Linear-Quadratic Programming

A control problem of the form

$$\sum_{t=0}^{T} \frac{1}{2} y_t' P_t y_t + p_t' y_t + \rho_t, u_t, M_t(u_t)$$

$$y_{t+1} = A_t y_t + B_t u_t, \quad y_0 = A_0 u_0$$

is called an Extended Linear-Quadratic Program.

- Introduced in context of deterministic and stochastic control by Rockafellar and Wets in '90.
- Versatile, still retains attractive computational structure.
Theorem (B., Casey, Wets)

If $w_t$ and $v_t$ have a PLQ density

$$w_t \propto \exp[-\rho_t, W_t, M_t(y)] \quad v_t \propto \exp[-\rho_t, V_t, N_t(y)]$$

with $M, N \succeq 0$, then the smoothing MAP problem with dynamics

$$x_{t+1} = F_t x_t + w_t, \quad z_t = H_t x_t + v_t$$

is dual in the convex analytic sense to the control problem

$$y_t = F'_t y_{t+1} + H'_t u_t, \quad y_T = H'_T v_T$$

$$y_t \in W_t, \quad u_t \in V_t$$

with cost

$$\sum_{t=0}^{T} \frac{1}{2} y'_t M_t y_t + \frac{1}{2} u'_t N_t u_t - z'_t u_t$$
When $N > 0$, control objective becomes

$$\sum_{t=0}^{T} \frac{1}{2} y_t' M_t y_t + \frac{1}{2} (u_t - N_t^{-1} z_t)' N (u_t - N_t^{-1} z_t)$$
When $\mathcal{N} \succ 0$, control objective becomes

$$
\sum_{t=0}^{T} \frac{1}{2} y'_t M_t y_t + \frac{1}{2} (u_t - N_t^{-1} z_t)' N (u_t - N_t^{-1} z_t)
$$

- System state near 0
- Control follows trajectory $\{N^{-1}(z_t)\}_{t=T}^{0}$

This result generalizes the classical case, because the normal distribution is a PLQ density.
Introducing the Problem

Application to Nonparametric Estimation

Figure: Laplace Penalty

Figure: Nonparametric Estimation of Laplace Penalty $\rho$

Figure: Conjugate of Nonparametric Estimation $\rho^*$
Introducing the Problem

Performance of Laplacian MAP

Estimation of 100 states & 100 mts (Laplace Noise, Laplace Penalty)

- Actual State
- Estimated State
Introducing the Problem

Performance of Data-derived MAP

Estimation of 100 states & 100 mts (Laplace Noise, Nonpara Error)

- Blue dots: Actual State
- Red dots: Estimated State
Want to generalize classical estimation results to the case of PLQ case

Attractive computational results exist when noise is assumed to be normal.
  - General framework that allows for diverse range of distributions
  - Still contains structure similar to quadratic case

Estimation and Control Duality still holds in PLQ setting

How can this structure be used to our advantage while performing computations?
Thank you for your attention!

For more information see


at math.ucdavis.edu/~rbassett
References

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- *Duality of Linear Estimation and Control*. Simon and Stubberud.
- *General Duality Between Optimal Control and Estimation*. Todorov. ’08